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HOW TRISECTIONS OF THE ANGLE WERE TRANSMITTED FROM GREEK TO ISLAMIC GEOMETRY

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SUMMARIES

Two trisections of the angle were transmitted from Greek to Islamic geometry, one in the Arabic translation of the Lemmata *of (pseudo-?) Archimedes, and the other in a hitherto unpublished 9th-century treatise by Aḥmad ibn Mūsā, which contains a translation from another Greek source. This paper presents an edition of the Arabic text of the latter treatise, as well as an English translation and a commentary, in which the text is compared with Propositions 36-42 of Book 4 of the* Collection *of Pappus of Alexandria (4th century). Following this is a summary of the work of Thābit ibn Qurra on this trisection and an attempt to explain why some 10th-century Islamic geometers thought that the ancients had not been successful in trisecting the angle.*

ملخص
انتقل إلى العربية طريقتان لقسمة الزاوية بثلاثة أقسام متساوية - الأولى عن طريق الترجمة لكتاب "الماخوذات" المنسوب إلى أرشميدس ، و الثانية نجدها منقولة عن محرر يوناني آخر في مقال لأحمد بن موسى لم يسبق نشره و يرجع إلى القرن التاسع الميلادي . و يحتوى هذا البحث على تحقيق لهذا المقال الثاني مع ترجمة إنكليزية و تعليقات تتضمن المقارنة بين النص العربي و الأشكال ٣٦-٤٢ من المقالة الرابعة من مجموع مصنفات پاپس الإسكندراني (القرن الرابع للميلاد) و قد لخصت أيضًا مقالة ثابت بن قرة في قسمة الزاوية بثلاثة اقسام و حاولت أن أبين العلة في اعتقاد بعض الرياضيين الإسلاميين في القرن العاشر للميلاد أن القدماء لم يتوصلوا إلى هذه القسمة

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1. INTRODUCTION

Both in classical Greek geometry and in Islamic geometry the trisection of the angle was a fundamental problem. The Greeks trisected the angle using conic sections and other curves, and by means of the method known as *neusis* [Heath 1921-1922 1, 235-244], a term explained in Section 2. However, some Islamic geometers of the 10th century have said that they did not know of successful trisections by the ancients (see Section 7); so one might wonder whether trisections were in fact transmitted, and if so, in what way.

In fact two trisections have been transmitted. One of them is Proposition 8 of the *Book of Lemmata* of (pseudo-?) Archimedes. Here the transmission is clear, for the *Lemmata* were translated into Arabic by Thābit ibn Qurra al-Ḥarrānī (A.D. 836-901) [Sezgin 1974, 264-272]. Schoy [1926, 34-35] and Woepcke [1851, 117, note 2] have pointed out that traces of another Greek trisection are to be found in the work of Aḥmad ibn Mūsā ibn Shākir (one of the three brothers known as the Banū Mūsā who flourished in Baghdad in the middle of the 9th century [Sezgin 1974, 246-252]) and Thābit ibn Qurra. The only extant Greek text in which this trisection occurs consists of Propositions 36-42 of Book 4 of the *Mathematical Collection* of Pappus of Alexandria (henceforth referred to as $\mathcal{A}$ 4:36-42), which was written at the beginning of the 4th century. But as far as is known, Books 1-7 of the *Mathematical Collection* ($\mathcal{A}$) were not translated into Arabic, and hitherto no one has explained how this trisection was transmitted to Islamic geometry.

The purpose of this paper is to shed some light on this transmission by presenting the original Arabic text, an English translation, and a commentary on the hitherto unpublished extant part of the *Treatise by Aḥmad ibn Shākir on the Trisection of the Angle* (henceforth referred to as $\mathcal{B}$). It will be shown that $\mathcal{B}$ contains a translation of an unidentified Greek text closely related to $\mathcal{A}$ 4:37-40. However, $\mathcal{B}$ was probably not translated from $\mathcal{A}$ itself.

As a preliminary, the use of the constructions called *neusis* will be explained in Section 2. In Section 3 the mathematical contents of $\mathcal{A}$ 4:36-42 and $\mathcal{B}$ are described and compared. Section 4 contains the English translation of $\mathcal{B}$, and its language is compared with the language of $\mathcal{A}$ 4:37-40. The Arabic text of $\mathcal{B}$ can be found in the Appendix. In section 5 the relationship between $\mathcal{B}$ and $\mathcal{A}$ is discussed. Section 6 contains a discussion of a hitherto unpublished text, *The Trisection of the Rectilineal Angle, Composed by Thābit ibn Qurra*. This text contains the same trisection as $\mathcal{B}$ and $\mathcal{A}$ 4:36-42, but was much more widely known in Islamic geometry. Section 7 consists of a summary of some accounts of the history of the trisection of the angle, written by 10th-century geometers, who considered the trisection from the *Lemmata* as unacceptable and attributed the trisection common to $\mathcal{A}$ and $\mathcal{B}$ to Thābit ibn Qurra. Apparently for this reason they said that they did not know of successful trisections by ancient geometers.

This paper does not deal with trisections invented by Islamic geometers, such as that contained in the *Book on the Measurement of Plane and Solid Figures* (Liber trium fratrum de geometria), which was written by the Banū Mūsā. (see [Sezgin 1974, 246-248, 251-252, No. 1] for references to the literature on this trisection.)

The following system (the same as the one in [Toomer 1976, 33]) is used throughout to transcribe Greek and Arabic letters found in geometrical figures. The Arabic letter and the Greek letters are rendered successively in the transcription.

A = *'alif* = *A*, *B* = *bā'* = *B*, *G* = *jīm* = Γ, *D* = *dāl* = Δ,

E = *hā'* = *E*, *Z* = *zāy* = *Z*, *H* = *ḥā'* = *H*, Θ = *ṭā'* = Θ,

K = *kāf* = *K*, *L* = *lām* = Λ, The Arabic *yā'* is transcribed as *I*.

2. A NEUSIS AS A BASIC CONSTRUCTION AND AS A PROBLEM

A *neusis* (plural: *Neuseis*) or *verging* is the insertion of a straight-line segment of given length between two given straight or curved lines in such a way that the line segment verges toward a given point *P*; that is, *P* is on the produced part of the line segment.

Proposition 8 of the *Lemmata* contains an example of a *neusis* which is used to trisect an angle *ADE* (see Fig. 1): Draw a circle with center *D*, which intersects *DA* in *A* and *DE* in *E*. Produce *ED*. Insert a line segment *GB*, equal to the radius of the circle, between the circle and *ED* extended such that *GB* "verges" toward *A*. Now $\angle BDG = 1/3\ \angle ADE$. (Proof. $\angle ADE = \angle DAG + \angle G$. Since $AD = DB$ we have $\angle DAG = \angle DBA$, and since $DB = BG$ we have $\angle DBA = 2\ \angle G$. So $\angle ADE = 3\ \angle G = 3\ \angle BDG$. This proof is unlike the proof in the *Lemmata*; see [Ver Eecke 1960 2, 532-533].)

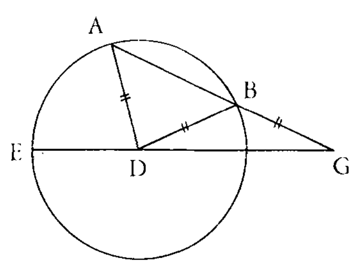


Figure 1

It seems that the author of this construction considered the insertion of *GB* as a legitimate geometrical operation, taking the *neusis* to be a basic construction which needs no further justification. In *A* 4:36-42 and *B* the trisection of the acute angle is also achieved by the insertion of a line segment of given length between two given lines, so that it verges toward a given point. But the texts also show how one can insert such a line segment using a construction by conic sections. Thus, here a *neusis* can no longer be considered a basic construction, but needs further justification; the construction of a *neusis* has now become a problem.

3. THE TRISECTION IN THE COLLECTION OF PAPPUS (*A*) AND IN THE TREATISE BY AHMAD IBN MŪSĀ (*B*)

The aim of this section is to give a mathematical analysis of the trisection as it occurs in *B* and in the printed Greek text of *A* 4:36-42. This printed Greek text is in [Hultsch 1965, 272-280]; a French translation is available in [Ver Eecke 1933, 210-217] and an English translation of *A* 4: Propositions 36, 38 is in [Bulmer-Thomas 1967-1968, 353-357]. Both in *A* and *B* the trisection consists of three parts, denoted by (i), (ii), and (iii). Part (i) is a construction of a hyperbola through a given point, with given asymptotes; such a hyperbola always exists and is uniquely determined. (I use the term *hyperbola* in the ancient sense, namely, a single branch of what is nowadays called a hyperbola.) Part (ii) is a construction of a *neusis* by means of a circle and a hyperbola. Part (iii) is the trisection proper.

Part (i) is found in *A* 4: Propositions 41-42 (see [Hultsch 1965, 276-280; Ver Eecke 1933, 214-217]). We no longer have the full text of *B*; part (i) is missing, although the available manuscripts say that Aḥmad explained (i) and "mentioned the proof of Apollonius" (see my translation in Section 4, Fol. 131b: lines 29-30). In *Conics* 2:4, Apollonius gives a construction of the hyperbola which is slightly different from (i) in *A* [Heiberg 1891 1, 198-201; Ver Eecke 1963, 121-122].

For the remainder of this section the arguments of *A* are rendered in Roman type, those of *B* in italics; the latter are given only when they are different from the former. I use notations, such as =, $\angle$, and $\square$ which do not exist in the Greek and Arabic texts. $\square$ *AB, GD* denotes the rectangle contained by *AB* and *GD*. The notation (X a/b:Y) means line Y of folio X, a or b. The geometrical figures are drawn as in *A*. My comments are in square brackets. The letters in the geometrical figures of *B* are the natural Arabic transcriptions of those in *A*; the only exception is the Θ in Fig. 2, which does not occur in *B*.

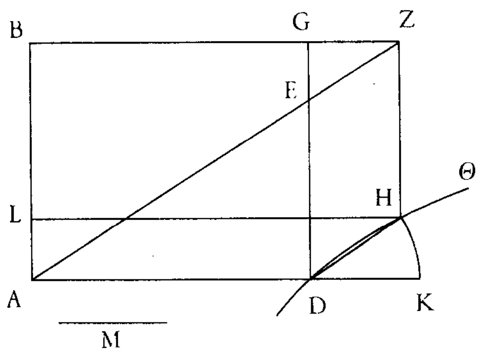


Figure 2

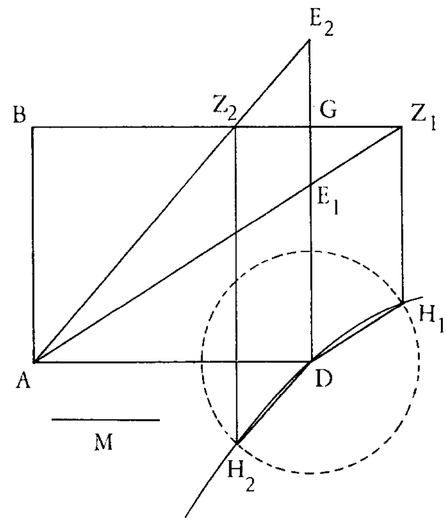


Figure 3

In (ii) the following problem is solved in $\mathcal{A}$ as well as in $\mathcal{B}$ (see Fig. 2): *Given:* a rectangle *ABGD* and a line segment *M*. *Required:* to construct a straight line *AEZ* such that its intercept *EZ* between *GD* and *BG* extended equals *M*. Thus *EZ* "verges" toward *A* and is equal to *M*.

[An analysis of the problem is presented in $\mathcal{A}$ 4:36, but not in $\mathcal{B}$, so I do not treat it in full. The synthesis is in $\mathcal{A}$ 4:37 and $\mathcal{B}$ (131b:30-132a:1):] *Solution:* Let *DK* = *M*. $\mathcal{B}$ *has: Produce AD toward K; let DK = M.* [In Fig. 2 note that in $\mathcal{A}$, too, *K* is on *AD* extended.]

Draw a hyperbola *DH*Θ through *D*, having asymptotes *AB* and *BG* [this is an application of (i)]. *Our manuscripts of* $\mathcal{B}$ *do not say that the hyperbola is DH*Θ *or that it has asymptotes AB, BG.* [This omission is most likely due to a scribal error, for the text does show some corruption at this point.]

Draw the circumference of a circle with center *D* and radius *DK*. Let this circumference intersect the hyperbola in *H*. $\mathcal{B}$ *has: "segment of a circle" instead of "circumference of a circle."* [$\mathcal{B}$ is more correct here, for a complete circumference would intersect the hyperbola in two points, H_1 and H_2, as indicated in Fig. 3, which is neither in $\mathcal{A}$ nor $\mathcal{B}$. This would lead to the construction of two straight lines, not only the desired line AE_1Z_1 such that $E_1Z_1 = M$, but also AZ_2E_2, which intersects *BG* in Z_2 and *DG* extended in E_2 such that $E_2Z_2 = M$. So $\mathcal{A}$ neglects the second point of intersection, and $\mathcal{B}$ assumes tacitly that the segment of the circle is drawn in such a way that it intersects the hyperbola only in H_1 in Fig. 3.] Draw HZ parallel to DG. $\mathcal{B}$ *adds: produce BG to meet HZ in Z.* [This addition is correct; it completes the definition of Z.] Draw AEZ. Now $EZ = M$.

Proof: Join HD, draw HL parallel to KA. We have $\square\, BZ, ZH = \square\, ZH, HL = \square\, GD, DA = \square\, BG, GD$. $\mathcal{B}$ *refers here to Conics 2:12.* [See [Heiberg 1891, 212-215; Ver Eecke 1963, 128-129]. *Conics* 2:12 is the following theorem: Let D, H be two points on a hyperbola with asymptotes AB, BG. Let $DG \parallel HZ$, $DA \parallel HL$ as in Fig. 4. Now $\square\, ZH, HL = \square\, GD, DA$.]

So $ZB:BG = GD:ZH$. [This follows from $\square\, BZ, ZH = \square\, BG, GD$.] But also $ZB:BG = GD:DE$. $\mathcal{B}$ *has* $ZB:BG = ZA:AE = GD:DE$ *instead of* $ZB:BG = GD:DE$. [This difference will be discussed below. Thus $GD:DE = ZB:BG = GD:ZH$], so $DE = ZH$; therefore $DEZH$ is a parallelogram. $\mathcal{B}$ *does not say that DEZH is a parallelogram, but has: EZ and DH join DE and ZH.* The conclusion is the same in $\mathcal{B}$ and $\mathcal{A}$, namely, $EZ = DH$. [But H is on the circle] so $DH = DK = M$. Therefore $EZ = M$, which had to be proved.

The difference between (b) $ZB:BG = ZA:AE = GD:DE$ in $\mathcal{B}$ and (a) $ZB:BG = GD:DE$ in $\mathcal{A}$ presupposes two different lines of mathematical reasoning. Statement (b) supposes the following argument: $ZB:BG = ZA:AE$ because $AB \parallel EG$ and $ZA:AE = GD:DE$ because $ZG \parallel AD$. But Pappus in his analysis in $\mathcal{A}$ 4:36 does not argue in this way to justify (a). (Actually (a) does not appear in this analysis, but only the equivalent equality of areas $\square\, BZ, ED = \square\, BG, GD$ is present.) Here Pappus reasons as follows (see Fig. 5): Suppose we have constructed AEZ such that $EZ = M$. Draw $ZH \parallel ED$, $DH \parallel EZ$, and let them intersect in H. Now $ZH = DE$, so $\square\, BZ, ZH = \square\, BZ, ED = \square\, BG, GD$ so H is on a hyperbola, etc.

Without further explanation, Pappus says $\square\, BZ, ED = \square\, BG, GD$. This equality results from Euclid's *Elements* 1:43, as was pointed out by modern commentators such as Hultsch [1965, 273] and Bulmer-Thomas [1967-1968, 355]: Complete the rectangle $BAXZ$, and through

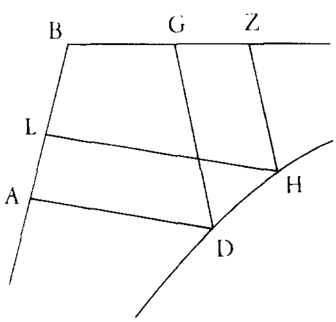


Figure 4

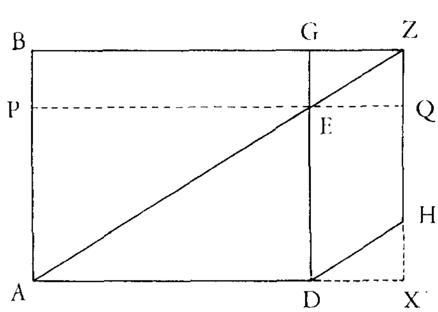


Figure 5

E draw *PQ* parallel to *BZ*. (Dotted lines do not appear in *A*.) *AZ* is a diagonal of the rectangle *BAXZ*; now *Elements* 1:43 states that *EQXD* and *BGEP* are equal [Heath 1956 1, 340-341]. Therefore *PQXA* and *BGDA* are equal and $\square BZ, ED = \square BG, GD$. I conclude that Pappus had an equality of areas in mind when he wrote (a); however, (b) represents an argument from similarity. It seems to me that here the argument of *B* is more straightforward than the argument of *A*.

To summarize: *B* and *A* 4:37 agree to a remarkable extent, although there are some differences between the mathematical arguments. *B* presents a version of (ii) which is more correct, more straightforward, and more in harmony with the geometrical figure.

In (iii), *B* and *A* 4:38 give the trisection of the acute angle *ABG* (see Fig. 6). Drop a perpendicular *AG* to *BG*. Complete the parallelogram *AGBZ*. [It is striking that both *A* and *B* speak of a parallelogram, although *AGBZ* is a rectangle.] Construct a line *BDE* such that its intercept *DE* between *AG* and *ZA* extended equals twice *AB*. [This is an application of (ii); *AGBZ* is a rectangle, and twice *AB* is a known length.] Now $\angle EBG = (1/3)\ \angle ABG$.

Proof. Bisect *ED* in *H*; draw *AH*. Now $DH = HA = HE$. *B adds: because EAD is a right angle.* So $DE = 2AH$, but $DE = 2AB$, therefore $AH = AB$. *B omits* $DE = 2AB$. It follows that $\angle ABD = \angle AHD$. Also, $\angle AHD = 2\ \angle AED$, and [since $AE \parallel BG$], $\angle AED = \angle DBG$; so $\angle ABD = 2\ \angle DBG$.

If we bisect $\angle ABD$, then $\angle ABG$ will be trisected. *B omits this last sentence.*

A 4:39 is a trisection of the right angle; *B remarks simply that this is "easy"* (132a:6).

A 4:40 and *B* have the same trisection of the obtuse angle *ABG* (see Fig. 7): Draw *BD* perpendicular to *BG*. $\angle DBG$ is a right angle, and *DBA* is acute; so we can construct *BZ* and *BE* such that $\angle DBZ = (1/3)\ \angle DBG$, $\angle EBD = (1/3)\ \angle ABD$. So $\angle EBZ = (1/3)\ \angle ABG$. If we apply an angle equal to $\angle EBZ$ to both arms of $\angle ABG$, we shall have trisected $\angle ABG$.

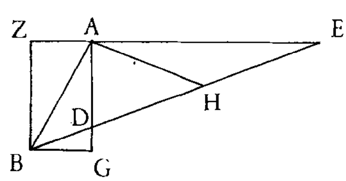


Figure 6

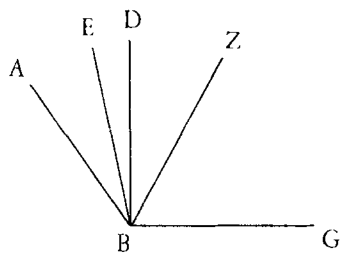


Figure 7

To summarize: there are few differences between the arguments of *B* and *A* in (iii).

I conclude this section by making some remarks on the origin of this trisection.

Pappus' own statements imply that he was not the author of the trisection in *A* 4:36-42. Before *A* 4:36 he states that the "earlier" geometers were unable to trisect the angle because they were not yet familiar with conic sections; but; *"Later, however, they trisected the angle by means of the conics, using in the solution the verging described below."* ([Hultsch 1965, 272, 15]; this translation is from [Bulmer-Thomas 1967-1968, 353].) The verging in question is that described in (ii).

The quotation suggests that this trisection was invented not long after the discovery of conics.

Another indication that this trisection is of early origin can be found in an unnecessary restriction in (ii), namely, that *ABGD* must be a rectangle. The construction and proof can be carried out in exactly the same way if one assumes simply that *ABGD* is a parallelogram, for *Conics* 2:12 is also valid if the angle between the asymptotes *AB* and *BG* of the hyperbola is not a right angle. Now one could argue that it is sufficient to prove (ii) for a rectangle *ABGD*, for in (iii), where one applies (ii), *AGBZ* is also a rectangle. However, I believe that the author of the trisection wished to remove unnecessary restrictions in (ii), for he chose *M* as an arbitrary line segment and did not assume $M = 2AD$, although we have $EB = 2AG$ in (iii).

The fact that (ii) is constructed only for a rectangle *ABGD* shows, in my opinion, that the author did not know a theorem as general as *Conics* 2:12, but only the following special case: Let P_1, P_2 be points on a hyperbola, whose asymptotes CQ, CR are at right angles. Draw P_1Q_1, P_2Q_2 parallel to CR, and P_1R_1, P_2R_2 parallel to CQ as in Fig. 8. Then the rectangles $CQ_1P_1R_1$ and $CQ_2P_2R_2$ are equal.

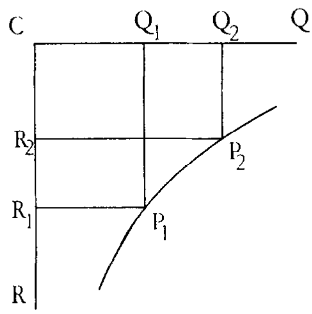


Figure 8

This property was already known to the reported discoverer of conic sections, Menaechmus (ca. 350 B.C.), who used it in his construction of two mean proportionals between two given lines (see [Heath 1921 1, 253-254; Bulmer-Thomas 1967-1968, 278-283]).

In my opinion, these considerations show that the trisection common to A and B was invented before Apollonius wrote his *Conics* (ca. 200 B.C.).

4. THE TEXT OF THE "TREATISE BY AḤMAD IBN SHĀKIR ON THE TRISECTION OF THE RECTILINEAL ANGLE"

My edition of B is based on the 13th-century manuscript Thurston 3, (131v:28-132r:8) preserved in the Bodleian Library, Oxford. The other known manuscript which contains B is Marsh 720, from the same library; this manuscript is a 17th-century copy of Thurston 3. Since this copy contains many additional corruptions, I did not use it for my edition. The manuscripts do not contain the full text of B. As was mentioned above, a scribe omitted the section on the construction of the hyperbola.

The geometrical figure in B pertaining to (ii) is the mirror image of the figure in A, but the other geometrical figures in B occur in the same form in A.

The edited Arabic text of B can be found in the Appendix. The numbers in the Arabic text and English translation are the numbers of the lines in the manuscript.

A literal translation of the extant part of B follows. I have corrected corruptions as indicated in the Appendix and have supplied missing parts of the text in angular brackets, < >. Words I have added are in square brackets and the notation (X) signals the beginning of line number X.

(131b:28) Treatise by Aḥmad ibn Shakir on the Trisection of the Rectilineal Angle

(29) He demonstrated first: Let AB, BG contain an angle B, let a point D be between them. We wish to draw a hyperbola through D with asymptotes AB, BG. (30) He mentioned the proof of Apollonius.

Then he said: Let AG have parallel sides and right angles [Fig. 9]. Let BG be extended in a straight line. We wish to draw from (31) A a line to BG extended, such that the part of it which falls between GD and BG extended is as the assumed line M.

(32) We produce AD toward K and make DK as M. We draw a hyperbola through D <having asymptotes AB and BG>, and, with center D and radius (33) DK a segment of a circle KH, which intersects the [conic] section in H. We draw from H a line parallel to DG, (34) on which are H, Z. We produce BG till it meets it [HZ] in Z. We draw ZA. Then EZ is as M. For we join (35) DH, and draw HL parallel to KA. Then ZH by HL, which is equal to BZ by ZH, is as GD (36) by DA, which is equal <to BG by GD>, as is demonstrated in [proposition] 12 of [book] 2 of (37) the *Conics* [of Apollonius]. So ZB to BG, which is equal to ZA to AE and [is equal to] GD to DE, is as GD to ZH (132a:1) <so ZH is> equal to ED. But they were connected by EZ, DH. So EZ is as DH, and equal to DK, which is as M.

Having completed these preliminaries, we demonstrate how we divide the angle ABG (first, let it be (2) acute) into three equal parts [Fig. 10]. From the point A on AB we drop the perpendicular AG to BG. We complete the parallelogram GZ. We produce ZA toward E (3) and [draw] BE such that ED is twice AB. Then [angle] EBG is one-third of [angle] ABG. For we bisect ED in H, and we draw (4) HA. Then DH, HA, HE are equal because [angle] EAD is a right angle. So ED is twice HA. Therefore AB is as AH, so [angle] ABH is as [angle] AHD. (5) But [angle] AHB is twice [angle] E, which is equal to [angle] DBG. So [angle] ABD is twice [angle] DBG. This is what was required.

If the angle is (6) [a] right [angle] it [the proof] is easy.

If the angle is obtuse, as ABG, then we draw the perpendicular BD [Fig. 11]. We make [angle] DBZ one-third of the right [angle] DBG (7), and [angle] EBD one-third of the acute [angle] ABD. If we apply to both AB, BG (8) an angle as EBZ [meaning that one inserts an angle equal to EBZ twice into ABG], we shall have divided ABG into three parts. This is what was required. [In the margin of the manuscript are the words: "And in another way. We bisect the obtuse angle; then half of it is acute. Then we trisect half of it. Two-thirds of it are one-third of the obtuse [angle]."]

So that the languages of $\mathcal{B}$ and $\mathcal{A}$ can be compared, three quotations (and their translations), where the text of $\mathcal{A}$ is almost identical with that of $\mathcal{B}$, are given below:

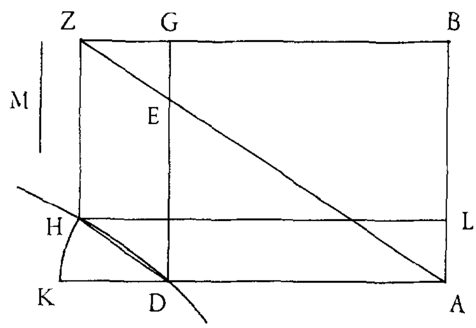


Figure 9

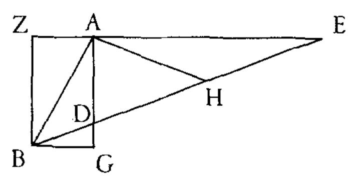


Figure 10

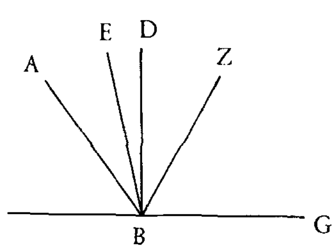


Figure 11

1. *B* 131b:34-36. *A* has:

Ἐπεζεύχθω γὰρ ἡ *HΔ* καὶ τῇ *KA* παράλληλος ἤχθω ἡ *HΛ*. τὸ ἄρα ὑπὸ *ZHΛ*, τουτέστιν τὸ ὑπὸ *BZH*, ἴσον ἐστὶν τῷ ὑπὸ ΓΔ*A*, τουτέστιν τῷ ὑπὸ *B*Γ ΓΔ. [Hultsch 1965, 274, 11-13]

For let HD be joined and let HL be drawn parallel to KA. The [*rectangle*] *contained by ZHL, that is, the* [*rectangle*] *contained by BZH, is equal to the* [*rectangle*] *contained by GDA, that is, the* [*rectangle*] *contianed by BG, GD.*

2. *B* 132a:3-4. "For we bisect ... equal." *A* has:

Τετμήσθω γὰρ ἡ *E*Δ δίχα τῷ *H*, καὶ ἐπεζεύχθω ἡ *AH*. αἱ τρεῖς ἄρα αἱ Δ*H HA HE* ἴσαι εἰσιν. [Hultsch 1965, 276, 7-8]

Let ED be bisected in H, and let AH be drawn. The three [*lines*] *DH, HA, HE are equal.*

3. *B* 132a:7-8. "If we apply ... parts." *A* has:

<ἐὰν δὲ τῇ ὑπὸ> *EBZ* ἴσην συστησώμεθα προς ἑκατέραν τῶν *AB*Γ, τρίχα τεμοῦμεν τὴν δοθεῖσαν γωνίαν. [Hultsch 1965, 276, 28-31]

If we apply an [*angle*] *equal to the* [*angle*] *contained by EBZ to both* [*arms*] *of ABG, we shall have trisected the given angle.*

(<ἐὰν δὲ τῇ ὑπὸ> is an addition made by Hultsch to the Greek text, which is thus confirmed by *B*.)

In these three examples *B* can be considered as a literal translation of the text in *A*. The only exceptions are in example 1, where *A* has *HD*, *B* has *DH*, in example 2, where *A* has *HA*, *B* has *AH*, and in example 3, where *A* adds the word "given."

Note that *B* shortens some geometrical expressions in a way which is unusual in Arabic, but normal in Greek. "The angle *ABG*" is normally rendered in Greek by

ἡ ὑπὸ *AB*Γ,

and not by

ἡ ὑπὸ τῶν *AB BΓ* περιεχομένη γωνία.

B has "*ABG*" instead of "zāwiya *ABG*." In example 1, *A* shortens "the rectangle contained by *ZHL*" to

τὸ ὑπὸ *ZHΛ*.

Here *B* has "*ZH fī HL*," probably to avoid confusion with "*ZHL*," which would mean "the angle *ZHL*." Some typical Arabic expressions for the rectangle contained by *ZHL* are "*saṭḥ ZHL*," "*saṭḥ ZH fī HL*," and "*ḍarb ZH fī HL*." Thus *B* is more concisely formulated than most other geometrical texts. This conciseness of *B* and the close resemblance between the texts in *B* and *A* as illustrated in the examples above show clearly that *B* is in fact a translation of a Greek text, which must be related to *A*.

As might be expected there are differences in the Greek and Arabic phrasing. Where *A* has a verb in the passive voice (as in example 1, "let *HD* be joined"), *B* often has the active ("we join *DH*") in 131b:34-35). Such differences do not affect the basic meaning of the text.

In the following example the texts of *A* and *B* are different:

4. *B* 132a:2-3 has: "We produce *ZA* toward *E* and [draw] *BE* such that *ED* is twice *AB*." *A* has:

> *Let ZA be produced to E, and inasmuch as GZ is a right-angled parallelogram, let the straight line ED be placed between EA, AG so as to verge toward B and be equal to twice AB; that this is possible has been proved above* ([Hultsch 1965, 274:21-276:4]; the translation is found in [Bulmer-Thomas 1967-1968, 355].)

5. THE RELATION BETWEEN *A* AND *B*

In the preceding section it was shown that *B* is essentially a translation of a Greek text. It is probable that its translator, Aḥmad ibn Mūsā, added the reference to the *Conics* in *B* 131b:36; this reference does not occur in *A* 4:37, for the preface to the Arabic translation of the *Conics* says that Aḥmad ibn Mūsā also went through the *Conics* and interpolated references wherever Apollonius tacitly used theorems that he had proved earlier in the *Conics* [1] (these references can still be found in the translation of *Conics* 5-7 by Ver Eecke [1963]). But it is also probable that Aḥmad ibn Mūsā made no further additions to $\mathcal{B}$; for otherwise it would not be as concisely formulated as it is now but would be written in the style of the introduction to the translation of the *Conics*, which contains theorems of the Banū Mūsā themselves [2].

Thus one wonders from which Greek text $\mathcal{B}$ was translated. It is very difficult to believe that $\mathcal{B}$ was translated from $\mathcal{A}$ 4. Serious objections arise from the difference between some mathematical arguments in $\mathcal{B}$ and $\mathcal{A}$, especially between those called (a) and (b) in Section 3, and from textual differences as in example 4 of the preceding section.

Further, if the Banū Mūsā had had a copy of $\mathcal{A}$ 4, it is probable that they would have translated more of it into Arabic. $\mathcal{A}$ 4 contains many other propositions that would have been of interest to the Banū Mūsā, such as Propositions 43 and 44, which give two other trisections of the angle; however, other traces of $\mathcal{A}$ 4 have not yet been found in Islamic geometry and, therefore, I conclude that $\mathcal{B}$ is not a translation of a fragment of $\mathcal{A}$, but a translation of another text *X*, which must be related to $\mathcal{A}$.

I have been unable to identify *X*. It may be related to $\mathcal{A}$ in either of the following ways:

1. The author of X, written in late antiquity, copied the trisection from $\mathcal{A}$ *4:36-40, omitted the analysis in* $\mathcal{A}$ *4:36, and revised some details of* $\mathcal{A}$ *4:37 but left the rest of the text unchanged.*

2. X is not dependent on $\mathcal{A}$*, but on a geometrical work Y written before Pappus' time. Pappus may have copied the trisection from Y but made some changes; or, perhaps, Pappus had access only to a damaged and corrupt copy of Y and so had to reconstruct several details in the reasoning. An analysis such as* $\mathcal{A}$ *4:36 did not figure in Y but was added by Pappus. It is not necessary to assume that Y was written by the author of the trisection; Y could have been written at any time between 200 B.C. and A.D. 300.*

I believe that the second of these two hypotheses is the more plausible, because in (ii) $\mathcal{B}$ is more in harmony with the geometrical figure and more straightforward than $\mathcal{A}$, suggesting that $\mathcal{B}$ depends not on $\mathcal{A}$ but on a work Y, as described in the second hypothesis. Such a work could have presented the trisection in a way that was more closely related to the original form in which it was worked out.

6. THE *TRISECTION OF THE RECTILINEAL ANGLE, COMPOSED BY THĀBIT IBN QURRA*

The subject of this section is an unpublished text, called *The Trisection of the Rectilineal Angle, Composed by Thābit ibn Qurra,* henceforth referred to as $\mathcal{C}$. A manuscript of $\mathcal{C}$, copied in A.D. 969, is extant in the Bibliothéque Nationale in Paris (Fonds Arabe 2457/45, 192b-194a [De Slane 1883, 433; Sezgin 1974, 271, No. 16]).

Thābit's trisection of the angle in $\mathcal{C}$ is similar to the trisections described in $\mathcal{B}$ and $\mathcal{A}$. Also, $\mathcal{C}$ is divided into three propositions, corresponding to (i), (ii), and (iii) in Section 3.

Part (i), the construction of a hyperbola, is a literal translation of *Conics* 2:4, although there is no reference to the *Conics* in our text of $\mathcal{C}$.

Next Thābit constructs the *neusis* (Fig. 12) of (ii) by means of a circle and a hyperbola. (Figures 12 and 13 are drawn as they occur in the manuscript, except that in the manuscript Fig. 12 is drawn so that *AE*, *TZ* and *DG* intersect at *H*.) *ABGD* is a parallelogram; *I* is a given line segment. *Required:* to construct a line *AHE* such that *HE* = *I*, as in Fig. 12. The construction and proof follow exactly the same lines as $\mathcal{B}$. However, Thābit removes the unnecessary restriction that *ABGD* be a rectangle.

Parts (i) and (ii) are unconnected from a terminological point of view. In (i) Thābit uses the term "*al-qaṭ al-zā'id*" [the exceeding section] to translate "hyperbola"; in (ii) the hyperbola is called "*al-qaṭ al-musammā ūbarbūlā*" [the section called hyperbola].

In (iii), Thābit trisects the angle *ABG* (in Fig. 13, *ABG* must be an acute angle, although the text does not say so). He drops a perpendicular *AD* to *BG*, draws through *A* a line parallel to *BG*, and then draws *BZE* such that *ZE* = 2*AB*.

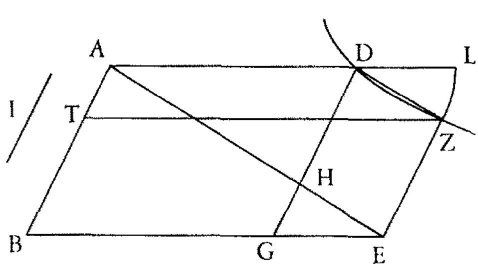


Figure 12

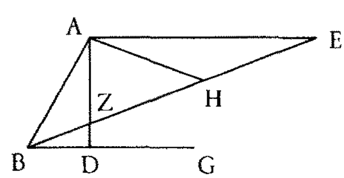


Figure 13

This construction is an application of (ii), but Thābit fails to make any reference to a parallelogram as mentioned in (ii). This suggests that Thābit did not invent the division into (i), (ii), and (iii) himself, but took it over from another source. Thābit's proof that $\angle ZBD = (1/3) \angle ABD$ is similar to $\mathcal{B}$.

Thābit's style does not have the extreme conciseness of $\mathcal{B}$. Thus, where $\mathcal{B}$ refers to geometrical objects simply as "*A*," "*ABE*," Thābit would say: "the point *A*," "the angle *ABE*." Thābit gives more arguments to support his statements than $\mathcal{B}$. For example, in (ii) Thābit states that *DL* and *DZ* are equal (Fig. 12) "because they are radii in a circle." He also quotes *Conics* 2:12 in full before proceeding to the consequence, $\square$ *AD*, *DG* = $\square$ *TZ*, *ZE* in (ii).

There is no reason to suppose that (ii) and (iii) in $\mathcal{C}$ are translations from the Greek [3]. A contra-indication is the use of the letter *I* in (ii), which would be the transcription of the Greek "I," if (ii) were a translation. However, Greek geometrical works seldom use "I" to indicate points and lines in figures. So it is probable that Thābit's $\mathcal{C}$ is heavily dependent on $\mathcal{B}$. This is not surprising, for we know that Thābit was a pupil of the Banū Mūsā and translated *Conics* 5-7 under their supervision. It seems plausible that Thābit wrote $\mathcal{C}$ to replace $\mathcal{B}$, which was difficult to understand because of its conciseness.

7. CONCLUSION

A brief outline of the rest of the history of Greek trisections in Islamic geometry follows. This topic has been treated in more detail in an earlier publication, in which appropriate references and quotations in Arabic are given [Hogendijk 1979].

No ancient geometer's name was attached to the trisection common to $\mathcal{A}$ and $\mathcal{B}$, and, therefore, its origin was forgotten. Abū Ja^cfar Muḥammad ibn al-Ḥusayn al Khāzin [Sezgin 1974, 298-299, 305-307] [4] said, in the 10th century, "And concerning the first question--that is, the (tri-)section of the angle--I did not hear that any one of them (the ancients) made any contribution to its solution in the geometrical way, with which (contribution) the solution can be constructed" [5]. His younger contemporary Al-Sijzī [Sezgin 1974, 329-334] was of the same opinion (see [Woepcke 1851, 117-118]). Both geometers asserted that Thābit ibn Qurra was the first to trisect the angle. Al-Sijzī gave a "lemma of Thābit ibn Qurra," which is (iii) in $\mathcal{C}$ and Abū Ja^cfar stated that Thābit made use of the hyperbola. So Al-Sijzī attributed the trisection common to $\mathcal{A}$ and $\mathcal{B}$ to Thābit ibn Qurra. This shows that the influence of $\mathcal{C}$ was greater than the influence of $\mathcal{B}$.

The trisection by means of a *neusis* in the *Lemmata* was not considered as a legitimate geometrical construction--at least not by Al-Sijzī, who described it in his *Treatise on the Division of the Angle into Three Equal Parts* [Sezgin 1974, 331, No. 7], part of which was translated into French by Woepcke [1851, 117-125]. Al-Sijzī called it a "lemma [invented] by one of the ancients, using the ruler and the moving geometry (*al-handasa al-mutaḥarrika*), but we have to solve it by fixed geometry (*al-handasa al-thābita*)" (see [Woepcke 1851, 120]). The legitimate geometrical constructions belonging to this fixed geometry were those using immovable straight lines, circles, and conic sections.

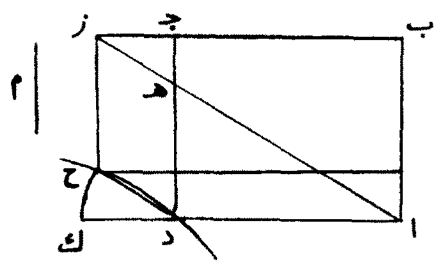
APPENDIX

Arabic text of the "Treatise by Aḥmad ibn Shākir on the trisection of the rectilineal angle".

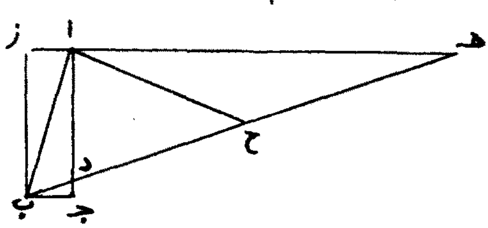
(131b:28) قول لأحمد بن شاكر في تثليث الزاوية المستقيمة الخطين

(29) بين أولا أنه إذا كان اب بجـ محيطين بزاوية ب
و نقطة د فيما بينهما ، فنريد أن نرسم على د قطعا زائدا
لا يقع عليه اب بجـ . (30) فذكر برهان أبلونيوس .
ثم قال : إذا كان اجـ متوازى الأضلاع قائم الزوايا و أخرج
بجـ على استقامة ، فنريد أن نخرج من (31) ا خطا إلى
بجـ المخرج حتى يكون الواقع بين جـد و بجـ المخرج
كخط م المفروض . (32) فنخرج اد إلى ك ، و نجعل دك
كم . و نخط على د قطعا زائدا < لا يقع عليه اب بجـ > ،
و على مركز د و ببعد (33) دك قطعة دائرة كح تقطع
القطع على ح . و نخرج من ح خطا موازيا ل دجـ (34)
عليه ح ز ، و نخرج بجـ حتى يلقاه على ز . و نصل زا .
فهز كم .

لأنا نصل (35) د ح و نخرج ح ل موازيا ل ك ا .
ف ز ح في ح ل المساوى ل ب ز في ز ح ك جـ د (36) في
د ا المساوى < ل ب جـ في جـ د > ، كما تبين في يب من ب
من (37) المخروطات . ف ز ب إلى ب جـ المساوية ل ز ا إلى ا هـ
و جـ د إلى د هـ ك جـ د إلى ز ح . (132a:1)
<فيكون ز ح> مساويا
ل هـ د ، و قد وصل بينهما
ب هـ ز ، د ح ، ف هـ ز
ك د ح و مساو ل د ك
الذى ك م .



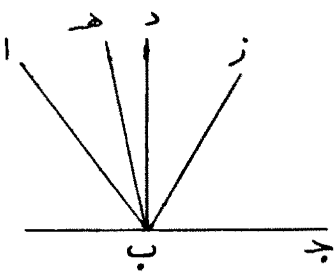
و إذ قدمنا ذلك فنبين كيف نقسم زاوية ا ب جـ ،
و لتكن أولا (2) حادة ، بثلاثة أقسام متساوية .
فنخرج من نقطة ا
على ا ب عمود ا جـ
على ب جـ ، ونتمم
متوازى الأضلاع جـ ز .
فنخرج ز ا إلى هـ
(3) و ب هـ حتى
يكون هـ د ضعف ا ب .
ف هـ ب جـ ثلث ا ب جـ .



لأنا ننصف هـ د على ح و نصل (4) ح ا ،
ف د ح ، ح ا ، ح هـ متساوية لأن هـ ا د قائمة .
ف هـ د ضعف ح ا ، ف ا ب ك ا ح ، ف ا ب ح ك ا ح د .
(5) لكن ا ح ب ضعف هـ المساوية ل د ب جـ .
ف ا ب د ضعف د ب جـ و هو المراد .

و إن كانت الزاوية (6) قائمة فسهل .

و إن كانت منفرجة كـ ا ب جـ ، فنخرج عمود ب د . و نجعل
د ب ز ثلث د ب جـ القائمة ،(7) و هـ ب د ثلث ا ب د
الحادة . فإذا أضفنا إلى
كل واحد من ا ب ب جـ
(8) زاوية كـ هـ ب ز ،
فنكون قد قسمنا
ا ب جـ بثلاثة أقسام ،
و هو المراد .



CRITICAL APPARATUS

Approximately half of the diacritical marks are missing from the manuscript. I have supplied those that are missing. I have also supplied all commas and full stops.

In the following cases my reading differs from the manuscript (I render my reading to the left of the colon, and the text of the manuscript to the right).

131b:28 بثلثة : بثلاثة , 29 و نرد : فنريد .

30 ونرد : فنريد . 32 زائد : زائدا A has καὶ γεγράφθω διὰ μὲν τοῦ Δ περὶ ἀσυμπτώτους τὰς ΑΒΓ ὑπερβολὴ ἡ ΔΗΘ [Hultsch 1965, 274, 5-7]

I have supplied لا يقع عليه ا ب ب جـ , since a ط is not extant in the figure in B.

36 I have supplied ل بج في جد

where the manuscript repeats ل زح في حل المساوى ل بز في زح

37 : Near و there is a blot in the manuscript which does not belong to the text; the manuscript reads مو .

<u>132a</u>:1 I have supplied . قد بصل : قد وصل . فيكون زح

2 فقسم • فقسم فنخرج . فنخرج · ثلثه : بثلاثة is crossed out.

3 · بصف : ننصف

8 . سلثه : بثلاثة . يكون : فنكون

Beside 132a:6-8 the manuscript has in the margin:

و بوجه آخر ننصف الزاوية المنفرجة فيكون نصفها حادة ثم نثلث نصفها فيكون ثلثاها ثلث المنفرجة

The geometrical figure which corresponds to 131b:30-132a:1 is drawn in the manuscript such that the three lines لح ، دج ، از all pass through the point ه

ACKNOWLEDGMENTS

I thank the Bodleian Library, Oxford, and the Bibliothèque Nationale, Paris, for providing microfilms of Arabic manuscripts; and the Curators of the Bodleian Library, Oxford, for permission to publish an edited text and English translation of MS. Thurston 3, fol. 131v:28-fol. 132r:8. I am grateful to Professor A. I. Sabra, Cambridge, Massachusetts, who translated the summary of this paper into Arabic; to Miss S. M. McNab, Utrecht, Dr. J. L. Berggren, Vancouver, and Dr. D. King, New York, for numerous linguistic and stylistic improvements; and to Professor G. J. Toomer, Providence, Rhode Island, for a number of valuable suggestions.

NOTES

1. See lines 14-15 of the photocopy of fol. 224a of MS. Aya Sofia 4832 in [Terzioglu 1974]; Terzioglu's translation is not accurate here.

2. I consulted MS. Oxford, Bodleian Library, Marsh 667, fols. 3b-7a. See also [Terzioglu 1974], the part of the photocopy of MS. Aya Sofia 4832, not translated by Terzioglu.

3. My suggestion in [Hogendijk 1979] that C is entirely a translation of some Greek source is false.

4. Abū Ja^{c}far al-Khāzin [Sezgin 1974, 298-299] and Abū Ja^{c}far Muḥammad ibn al Ḥusayn [Sezgin 1974, 305-307] are the same person (see [Sezgin 1978, 189]). Abū Ja^{c}far used the construction of (ii), as found in the *Trisection*, to transform the construction of two mean proportionals by means of the conchoid of Nicomedes into a construction by means of conic sections. In this application of (ii), *ABGD* is a parallelogram but not a rectangle. See [De Vaux 1898; Kohl 1922-1923, 186-189].

5. Fols. 82a-104b of MS. Oxford, Bodleian Library, Huntington 237, contain an extract of the "Islāḥ Kitāb al Makhrūṭāt" by Abū Ja^{c}far. Fol. 104b deals with the history of the trisection; the passage quoted is found in lines 8-10. See [Sezgin 1974, 307, 4].

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